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THE METHOD OF DISCRETE SINGULARITIES IN PLANE PROBLEMS OF THE THEORY OF ELASTICITY*

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Plane problems of the theory of elasticity are reduced to sets of singular integral equations for which a direct method of solution is developed, similar to the method of discrete vortices used in aerodynamics. Numerical solutions of a number of plane problems of the theory of elasticity are considered, stable numerical solutions are obtained, and their convergence is proved.

When solving problems of the theory of elasticity by reducing them to integral equations, the tendency usually was to get away from the singular integral equations (SIE), and to reduce them to regular integral equations of the first or second kind /1,2/. A similar situation occurs when solving other problems, for example, in electrodynamics /3/. It appeared, however, that numerical solutions of regular integral equations of the first kind on a computer were unstable. Regular integral equations of the second kind, obtained in the theory of elasticity, possess eigenfunctions /2/, and therefore their numerical solution on a computer by direct methods is also unstable. In view of these inconveniences in reducing the problems to regular integral equations, they are reduced to SIE, for which a stable method (the method of "discrete vortices" /4/) for their numerical solution has been developed.

Below, a similar approach is developed for solving plane problems of the theory of elasticity. These problems for bounded simply connected regions, whose boundary is a closed Liapunov curve, are reduced to SIE of the first kind with Hilbert kernels in complex conjugate functions. The conditions are obtained that ensure the uniqueness of the solution of these equations. The equations are solved numerically using the method of discrete singularities, which is a development of the method of discrete vortices. The idea of this method consists in exchanging the set of SIE for a set of linear algebraic equations in unknown functions with boundary points selected in some special way, and specially situated in relation to points at which the values of the required functions are found.

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The numerical solution of the second basic problem (the stresses are given on the boundary) is specifically considered for a circle for various loads, either continuous or concentrated at a finite number of points of application of the forces. Stable numerical solutions are obtained for these when there is one or more axes of symmetry, as well as when there is none, and the convergence of the solutions is proved. A new method is developed which is convenient for computer calculation for solving overspecified sets of linear algebraic equations that replace SIE.

1. The solution of plane problems of the theory of elasticity when there are no volume forces reduces to determining two analytic functions /2,5/ (e.g., φ, ψ), that satisfy the boundary condition

$$\varkappa_k \varphi(t) - \overline{t\varphi'(t)} - \overline{\psi(t)} = f_k(t), \quad t \in L, \ k = 1, 2 \tag{1.1}$$

on the contour L bounding the region D. If condition (1.1) is defined in displacements, then $k = 1, \varkappa_1 = \varkappa$ where the quantity $\varkappa$ is defined in /2/, and

$$f_1 = 2\mu(u + iv) \tag{1.2}$$

where μ is the shear modulus and u, v are the displacements. If condition (1.1) is specified in stresses, then $k = 2, \varkappa_2 = -1$ and

$$f_2 = -i\int_{s_0}^{s} (\sigma_{x\nu} + i\sigma_{y\nu})\, ds + c_2 \tag{1.3}$$

(the notation is given in /2/).

2. It is proposed to seek the analytic functions $\varphi(z), \psi(z)$ in the form

$$\varphi(z) = \frac{1}{2\pi i}\int_L \frac{\omega(t)\,dt}{t - z}, \quad \psi(z) = \frac{1}{2\pi i}\int_L \frac{\overline{c\omega(t)} - \bar{t}\omega'(t)}{t - z}\,dt, \quad z \in D \tag{2.1}$$

where the auxiliary function $\omega(t)$ is obtained from (1.1) in the form of the integral equation

$$\frac{\varkappa_k - c}{2}\omega(t) + \frac{\varkappa_k}{2\pi i}\int_L \frac{\omega(\tau)\,d\tau}{\tau - t} + \frac{c}{2\pi i}\int_L \frac{\omega(\tau)\,d\bar{\tau}}{\bar{\tau} - \bar{t}} + \frac{1}{2\pi i}\int_L \overline{\omega(\tau)}\, d\left(\frac{\tau - t}{\bar{\tau} - \bar{t}}\right) = f_k(t) \tag{2.2}$$

with the complex parameter c whose selection enables the problem to be reduced to various types of equations.

When $c = -\varkappa_k$ we obtain the Fredholm integral equation due to Muskhelishvili /5/.

$$\varkappa_k \omega(t) + \frac{\varkappa_k}{2\pi i}\int_L \omega(\tau)\, d\left(\ln\frac{\bar{\tau} - \bar{t}}{\tau - t}\right) + \frac{1}{2\pi i}\int_L \overline{\omega(\tau)}\, d\left(\frac{\tau - t}{\bar{\tau} - \bar{t}}\right) = f_k(t) \tag{2.3}$$

A direct solution of Eq.(2.3) by the method of mechanical quadratures is difficult due to the presence of an eigenfunction, which results in a degenerate set of linear algebraic equations and to unstable values of the unknown function.

Similar difficulties occur when the problem is reduced to other regular integral equations of the Muskhelishvili and Sherman and Lauricelli type /2,6/. Various methods were considered for eliminating this difficulty /2/, e.g. fixing ω at some points and eliminating the corresponding equations, and the use of the error of quadratic formulas to improve the structure of the algebraic equation.

When $c = 0$, Eq.(2.2) becomes a degenerate SIE of the second kind. A non-degenerate SIE of the second kind can be obtained, for instance, when $c = i$. Here we investigate the reduction to a SIE of the first kind, which is possible when $c = \varkappa_k$. Then, we obtain from (2.2)

$$\frac{\varkappa_k}{\pi i}\int_L \omega(\tau)\,\mathrm{Re}\left(\frac{d\tau}{\tau - t}\right) + \frac{1}{2\pi i}\int_L \overline{\omega(\tau)}\, d\left(\frac{\tau - t}{\bar{\tau} - \bar{t}}\right) = f_k(t) \tag{2.4}$$

Note that SIE (2.4) is an equation with a Hilbert kernel.

Indeed, let the region D be simply connected and the contour L that bounds it be smooth (a Liapunov one), i.e. its parametric equation $x = x(\eta), y = y(\eta)$ is such that $x(\eta), y(\eta), x'(\eta)$ and $y'(\eta)$ are 2π-periodic functions belonging to the Hölder class /7/. Note that

$$\mathrm{Re}\left(\frac{d\tau}{\tau - t}\right) = \frac{x'(x - x_0) + y'(y - y_0)}{(x - x_0)^2 + (y - y_0)^2}\,d\eta = \mathrm{ctg}\,\frac{\eta - \xi}{2}\,B(\eta, \xi)\,d\eta \tag{2.5}$$

$$\eta, \xi \in [0, 2\pi], \ \tau = x + iy, \ t = x_0 + iy_0, \ x_0 = x(\xi), \ y_0 = y(\xi)$$

As follows from /7/, the function $B(\eta, \xi)$ belongs to the Hölder class. It can be shown that it is periodic in η and ξ of periodic 2π and $B(\xi, \xi) = 0.5$. This implies that the kernel of (2.5) can be represented in the form of the sum of a Hilbert kernel and a regular kernel.

3. We shall investigate the properties of SIE (2.4) when L is a circle. On a circle SIE (2.4) has the form

$$\varkappa_k \int_0^{2\pi} \omega(\eta) \operatorname{ctg} \frac{\eta-\xi}{2} d\eta - i \int_0^{2\pi} \overline{\omega(\eta)} e^{i(\eta+\xi)} d\eta = 2\pi i f_k(\xi) \tag{3.1}$$

For the internal problem in stresses the vector of the external forces and the vector of the moment must vanish. Hence Eq.(3.1) must be supplemented by conditions on its right-hand side. The condition for the vector of the external forces to vanish requires uniqueness of $f_2(\xi)$, in whose presence it is automatically satisfied. For the function f_2 to be unique its periodicity in L is necessary, then

$$f_2(0) = f_2(L) \tag{3.2}$$

and in problems on a circle

$$f_2(\xi) = f_2(2\pi k + \xi); k = 0, \pm 1, \pm 2, \ldots \tag{3.3}$$

It is evident from (3.1) that condition (3.3) does not impose restrictions on the function ω.

The condition for the vector of the moment of the external forces to vanish /2/

$$\operatorname{Re} \int_L \overline{f_2(t)}\, dt = 0 \tag{3.4}$$

on a circle reduces, in relation to the function ω, using (3.1) for $\varkappa_k = \varkappa_2 = -1$, to the relation

$$-iR \int_0^{2\pi} [\overline{\omega(\eta)} e^{i\eta} + \omega(\eta) e^{-i\eta}]\, d\eta = \int_L \overline{f_2(t)}\, dt, \quad t = Re^{i\xi}$$

where on the left-hand side we have a purely imaginary expression for all ω; hence condition (3.4) does not impose any restrictions on ω.

Continuing the investigation of the problem on a circle, we see that the homogeneous equation (3.1), when $f_k \equiv 0$, has eigensolutions whose form is established, for example, from the representation of $\omega(\eta)$ in the form of a Fourier series. The eigenfunctions of the homogeneous equation (3.1) and (2.4) are the complex constant

$$\omega = a + ib \tag{3.5}$$

and, when $k = 2$ the function

$$\omega(\eta) = ia_1 e^{i\eta} \tag{3.6}$$

for the homogeneous equation (3.1), and for Eq.(2.4) the function

$$\omega(\tau) = ia_2\tau$$

To obtain a problem defined uniquely, it is necessary to proceed either as in /2/ or, as proposed below, the SIE must be solved together with some supplementary conditions that "will not pass" the eigenfunctions (3.5) and (3.6). The following conditions have this property:

$$\int_L \frac{\omega(\tau)}{\tau} d\tau = 0, \quad k = 1, 2; \quad \int_L \left[\frac{\omega(\tau)}{\tau^2} d\tau + \frac{\overline{\omega(\tau)}}{\bar{\tau}^2} d\bar{\tau}\right] = 0, \quad k = 2 \tag{3.7}$$

Sherman /2,6/ introduced as supplementary terms, the left-hand sides of Eqs.(3.7) in integral equations of the type (2.3) to ensure the uniqueness of the solution. On a circle conditions (3.7) are written as

$$\int_0^{2\pi} \omega(\eta)\, d\eta = 0, \quad k = 1, 2 \tag{3.8}$$

$$\operatorname{Im} \int_0^{2\pi} \overline{\omega(\eta)} e^{i\eta}\, d\eta = 0, \quad k = 2$$

and, consequently, ω in the form (3.5) does not satisfy the first of Eqs.(3.8), and in the form (3.6) it does not satisfy the second of Eqs.(3.8).

Besides, Eq.(3.1) has the property that the integral with respect to ξ from 0 to 2π on the left-hand side is zero, and consequently the right-hand integral of (3.1) must also be zero

$$\int_0^{2\pi} f_k(\xi)\, d\xi = 0 \tag{3.9}$$

which results in the requirement that the real, as well as the imaginary parts of function f_k must be zero. Condition (3.9), when solving the problem in stresses, determines the complex constant c_2 in (1.3), and when solving the problem in displacements, relation (3.9) is automatically satisfied for the function f_1 which is analytic in the region D and continuous on L. For the circle, condition (3.4) leads to the relation

$$\int_0^{2\pi} (f_R \sin\xi - f_I \cos\xi)\, d\xi = 0 \quad (f_2 = f_R + if_I) \tag{3.10}$$

For an arbitrary smooth contour L, by integrating (24) with respect to ξ, the parameter of the given contour L, we obtain the relation

$$\int_L f_k(\xi)\, d\xi = 0 \tag{3.11}$$

which is similar to condition (3.9) on the circle.

Thus the second basic problem for a region bounded by a closed smooth contour L reduces to the integral equation (2.4) and conditions (3.2),(3.4),(3.7), and (3.11) from which the function ω is determined, and when L is a circle, respectively to (3.1),(3.2), and (3.8)-(3.10), where the SIE has a singularity of the form $\operatorname{ctg}[(\eta-\xi)/2]$. This singularity of the SIE is preserved as shown above, in problems of arbitrary regions bounded by a smooth contour, but the regular part of the SIE changes.

4. We shall demonstrate the method of discrete singularities for Eq.(3.1) supplemented by conditions (3.3), and (3.8)-(3.10), setting $k=2$, i.e. $\varkappa_k = -1$. For this we separate the real and imaginary parts in (3.1) and (3.8). We obtain the set of equations (the integrals are taken from 0 to 2π)

$$\int \omega_R(\eta)\left[\operatorname{ctg}\frac{\eta-\xi}{2} - \sin(\eta+\xi)\right] d\eta + \int \omega_I(\eta)\cos(\eta+\xi)\, d\eta = 2\pi f_I(\xi) \tag{4.1}$$

$$\int \omega_R(\eta)\cos(\eta+\xi)\, d\eta + \int \omega_I(\eta)\left[\operatorname{ctg}\frac{\eta-\xi}{2} + \sin(\eta+\xi)\right] d\eta = -2\pi f_R(\xi)$$

$$\int \omega_R(\eta)\, d\eta = 0, \quad \int \omega_I(\eta)\, d\eta = 0$$

$$\int [\omega_R(\eta)\sin\eta - \omega_I(\eta)\cos\eta]\, d\eta = 0$$

where the subscript R denotes the real part of the respective function and I the imaginary part.

Let us, first, assume that the functions $f_R(\eta)$ and $f_I(\eta)$ belong to the Hölder class on $[0, 2\pi]$. Let the points η_i $(i = 1, \ldots, n)$, taken as points of a unit circle, divide the circle into n equal parts, and the points ξ_j be the midpoints of the arcs $\eta_i\eta_{i+1}$. We now replace the set of integral equations (4.1) for $k=2$ by the following set of linear algebraic equations:

$$\sum \omega_{nR}(\eta_i)\left[\operatorname{ctg}\frac{\eta_i-\xi_j}{2} - \sin(\eta_i+\xi_j)\right]\frac{2\pi}{n} + \tag{4.2}$$
$$\sum \omega_{nI}(\eta_i)\cos(\eta_i+\xi_j)\frac{2\pi}{n} + \beta_1 + \beta_3\cos\xi_j = 2\pi f_I(\xi_j)$$
$$\sum \omega_{nR}(\eta_i)\cos(\eta_i+\xi_j)\frac{2\pi}{n} + \sum \omega_{nI}(\eta_i)\left[\operatorname{ctg}\frac{\eta_i-\xi_j}{2} + \right.$$
$$\left.\sin(\eta_i+\xi_j)\right]\frac{2\pi}{n} + \beta_2 + \beta_3\sin\xi_j = -2\pi f_R(\xi_j)$$
$$\sum \omega_{nR}(\eta_i)\frac{2\pi}{n} = 0, \quad \sum \omega_{nI}(\eta_i)\frac{2\pi}{n} = 0$$
$$\sum [\omega_{nR}(\eta_i)\sin\eta_i - \omega_{nI}(\eta_i)\cos\eta_i]\frac{2\pi}{n} = 0; \quad j = 1, \ldots, n$$

where the summation is carried out over i from 1 to n, and β_1, β_2 and β_3 are regularizing factors /8/.

Without the unknown β_1, β_2 and β_3 (4.2) is overspecified (i.e. the number of equations is greater than the number of unknowns) and may generally be incompatible due to computational errors. However, even if it is compatible, it is difficult to find the three equations that must be rejected for it to become definable. The introduction of the unknowns β_1, β_2 and β_3 makes the system determinate and non-degenerate. The factors β_1, β_2 and β_3 approach zero as $n \to \infty$ if and only if conditions (3.9) and (3.10) are satisfied.

Summing the first n equations of (4.2), we obtain

$$-\frac{1}{2\pi}\sum_{i=1}^{n}\omega_{nR}(\eta_i)\frac{2\pi}{n}S_{si} + \frac{1}{2\pi}\sum_{i=1}^{n}\omega_{nI}(\eta_i)\frac{2\pi}{n}S_{ci} + \beta_1 + \frac{\beta_3}{2\pi}S_0 = \sum_{j=1}^{n} f_I(\xi_j)\frac{2\pi}{n} \tag{4.3}$$

$$\left(S_{si} = \sum_{j=1}^{n}\sin(\eta_i+\xi_j)\frac{2\pi}{n}, \quad S_{ci} = \sum_{j=1}^{n}\cos(\eta_i+\xi_j)\frac{2\pi}{n}\right.$$

$$S_0 = \sum_{j=1}^{n} \cos \xi_j \frac{2\pi}{n})$$

where the sums S_{si}, S_{ci}, S_0 are zero for odd n and approach zero as $n \to \infty$ for arbitrary n, since they approximate the respective integrals. By virtue of condition (3.9) the sum on the right-hand side of (4.3) also approaches zero as $n \to \infty$.

We thus find that (4.3) can be written in the form

$$\beta_1 + \alpha_1 \beta_3 = \alpha_2;\ \alpha_1, \alpha_2 \to 0 \text{ as } n \to \infty \tag{4.4}$$

Similarly, summing the following n equations, we obtain

$$\beta_2 + \alpha_3 \beta_3 = \alpha_4;\ \alpha_3, \alpha_4 \to 0 \text{ as } n \to \infty \tag{4.5}$$

If we now multiply the first n equations by $\cos \xi_j\ (j = 1, \ldots, n)$ and the second n equation by $\sin \xi_j\ (j = 1, \ldots, n)$, and add all $2n$ equations, taking condition (3.1) into account, we obtain

$$\alpha_5 \beta_1 + \alpha_6 \beta_2 + (1 + \alpha_7)\beta_3 = \alpha_8 \tag{4.6}$$

$$\alpha_5, \alpha_6, \alpha_7, \alpha_8 \to 0 \text{ as } n \to \infty$$

From Eqs. (4.4)-(4.6) we obtain the statement concerning β_1, β_2 and β_3 made above.

Note that the regularizing factors β_1, β_2 and β_3 can be introduced in other ways so that the condition for them to approach zero as $n \to \infty$ is satisfied, and the system remains non-degenerate. For instance, we can take β_1, $\xi_j \beta_2$ and $\xi_j^2 \beta_3$, in the first n equations, and β_1, $(2\pi + \xi_j)\beta_2$, $(2\pi + \xi_j)^2 \beta_3$ in the subsequent n equations.

The approximation of the integral with $\operatorname{ctg}[(\eta - \xi)/2]$ on the segment $[0, 2\pi]$ by the sums considered above follows from /8/. From the same paper it follows that for a characteristic SIE of the first kind with kernel $\operatorname{ctg}[(\eta - \xi)/2]$ Eqs. (4.2) can be similarly transformed into a set of linear algebraic equations for the regular Fredholm equation of the second kind, which has a unique solution, since Eqs. (4.1) have a unique solution. From this we obtain that Eqs. (4.2) are non-degenerate and their solution approaches the solution of integral equations (4.1). Then, if $f_R(\xi)$ and $f_I(\xi)$ belong to the class $H(\alpha)$ /5/ and n is an arbitrary positive integer, we have

$$\begin{gathered} |\omega_R(\eta_i) - \omega_{nR}(\eta_i)| \leqslant O_1(n^{-\alpha} \ln n) \\ |\omega_I(\eta_i) - \omega_{nI}(\eta_i)| \leqslant O_2(n^{-\alpha} \ln n) \end{gathered} \tag{4.7}$$

If however, n is odd and $f_R^{(r)}(\xi), f_I^{(r)}(\xi)$ belongs to the class $H(\alpha)$, it follows from /7/ that on the right-hand sides of inequalities (4.7) there are quantities of the order of $(\ln n)/n^{r+\alpha}$.

In the problem of loading by concentrated forces applied uniformly over a circle the functions $f_R(\xi)$ and $f_I(\xi)$ have discontinuities of the first kind at the points where the forces are applied. The calculation points ξ_j were located at these points, where the arithmetic mean of the one-sided limits for $f_R(\xi)$ and $f_I(\xi)$ were taken. The remaining points ξ_j divided the circle into equal parts, and the points η_i, $i = 1, \ldots, n$ were taken in the middle of these parts.

When the problem has axes of symmetry, the set of integral equations (4.1) can be transformed into a set of SIE of the first kind on a segment: some or all of the integral conditions on ω are then satisfied. When solving the set of SIE numerically along the segment, it is necessary to observe the following rule for the arrangement of the calculation points ξ_j and discrete singularities η_i on the segment of integration. This was obtained for one SIE of the first kind on the segment from heuristic considerations and numerical calculations in /4/ and was mathematically justified in /9/. Closest to the end of the segment at which the solution is unbounded is a discrete singularity, while nearest to the end of the segment at which the solution is bounded is a calculation point. In the case of (2.4) in the functions ω_R and ω_I, this rule must be applied in each equation with respect to that unknown function for which this equation is singular.

Thus, generally, it is necessary to take for ω_R and ω_I their proper sets of points η_{Ri} and η_{Ii}. Examples of numerical solutions of (4.2) are given in Figs.1 and 2.

A stable calculation and good convergence were obtained in all cases when the order of the system investigated was increased from 30 to 110. This was confirmed by comparison with the exact solution.

5. As an example, a continuous load was considered for which in Eqs. (4.1) the right-hand sides are the trigonometric functions $f_R = \sin \xi$, $f_I = \cos \xi$ (problem 1). In that case the functions $\omega_R = \sin \eta$, $\omega_I = \cos \eta$ are exact solutions of Eqs. (4.1). These values were compared with the results obtained for this problem using the method of discrete singularities. Solution of Eqs. (4.2) for all $n \geqslant 3$ (with $n_{\max} = 54$), when the general order N of Eqs. (4.2) was equal to $2n + 3$, yielded the following results: the values of ω_{nR}, ω_{nI} at the points at which they are defined are the same as the analytic solutions, and the regularizing factors $\beta_1, \beta_2, \beta_3$ are zero.

The numerical experiment has, thus, shown the absolute convergence of the method of discrete singularities for Eqs.(4.1) with the continuous right-hand side considered.

We then considered loading the circle by two concentrated loads applied at diametrically opposite points (problem 2, Fig.1). This problem

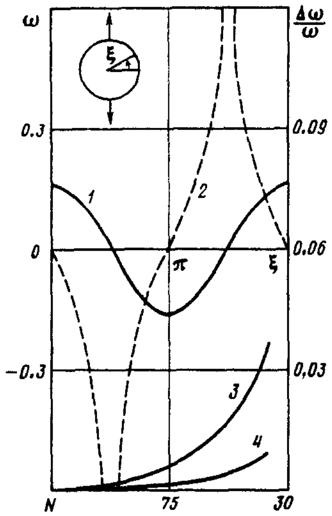

Fig. 1

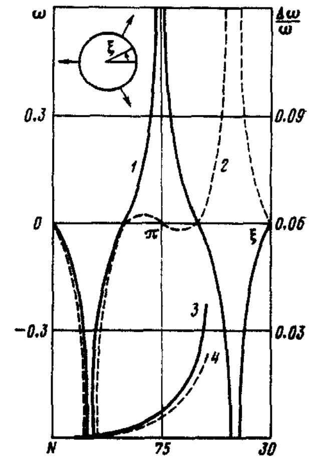

Fig. 2

was solved by the method of discrete singularities both by introducing three regularizing factors (Eqs.(4.12)), and by omitting one calculation point away from the points of discontinuity on the right-hand side of the system, which enables only one regularizing factor to be introduced. Both solutions are virtually the same, possessing good convergence with respect to $N = 2n + 3$. Calculations in which three regularizing factors are introduced are shown in Fig.1, where curves $1, 2, 3, 4$ correspond to the functions $\omega_R, \omega_I, \Delta\omega_I/\omega_I, \Delta\omega_R/\omega_R$, and $\Delta\omega_R, \Delta\omega_I$ are respectively the remainders of the functions ω_{nR}, ω_{nI} calculated for $n_{\max}$ and the current n.

The solution of problem 2 is even for the function $\omega_R(\eta)$ and odd for the function $\omega_I(\eta)$. Taking this into account, we reduce (4.1) to a set of SIE on the segment $[0, \pi]$

$$\sin\xi\left\{\int\omega_R(\eta)[(\cos\eta-\cos\xi)^{-1}+\cos\eta]\,d\eta+\int\omega_I(\eta)\sin\eta\,d\eta\right\}=-\pi f_I(\xi) \tag{5.1}$$

$$\cos\xi\int\omega_R(\eta)\cos\eta\,d\eta+\int\omega_I(\eta)[(\cos\eta-\cos\xi)^{-1}-\cos\xi]\sin\eta\,d\eta=\pi f_R(\xi)$$

$$\int\omega_R(\eta)\,d\eta=0$$

where the last two of the three integral conditions in (4.1) are now identically satisfied. The application of the method of discrete singularities to Eqs.(5.1) requires the rule defined above for the arrangement of calculation points and discrete singularties to be satisfied.

Comparison of the solutions of Eqs.(5.1) and (4.2) for problem 2 shows good agreement of the functions ω_{nR} and ω_{nI} for corresponding n, and also their rapid convergence with respect to n.

Problem 2 may be reduced to a set of SIE on the segment $[0, \pi/2]$, if the property of symmetry of ω_I and reverse symmetry of ω_R about the vertical axis is used.

Taking these properties into account, we can reduce Eqs.(5.1) to a set of SIE on the segment $[0, \pi/2]$ for problems on a circle with two perpendicular axes of symmetry

$$\sin\xi\left\{\int\omega_R(\eta)\cos\eta\,[(\cos^2\eta-\cos^2\xi)^{-1}+1]\,d\eta+\int\omega_I(\eta)\sin\eta\,d\eta\right\}=-\frac{\pi}{2}f_I(\xi) \tag{5.2}$$

$$\cos\xi\left\{\int\omega_R(\eta)\cos\eta\,d\eta-\int\omega_I(\eta)\sin\eta\,[(\cos^2\eta-\cos^2\xi)^{-1}+1]\,d\eta\right\}=\frac{\pi}{2}f_R(\xi)$$

Equations (5.2) do not contain supplementary integral conditions, since now all of them, taking the two axes of symmetry into account, are satisfied identically.

The loading of the circle by three equal concentrated radial forces applied at equal distances from one another (problem 3, Fig.2) was also considered. In this case, the solution of Eqs.(4.2) was derived using the method of discrete singularities, where the right-hand side of the equations was determined taking condition (3.9) into account. The solution and its convergence with respect to N are shown in Fig.2, where the notation for the curves is the same as in Fig.1. With the obvious selection of the origin of the coordinate ξ, this problem has a horizontal axis of symmetry relative to which the function ω_R is even and ω_I odd. This enables problem 3 to be reduced to Eqs.(5.1).

6. If the first basic problem is considered on the circle, i.e. the displacements are specified at the boundary, it is necessary to consider another system instead of Eqs.(4.1), since the function (3.6) is not an eigenfunction of the homogeneous equation (3.1) when $\varkappa_k = \pm 1$ (integration is carried out in the limits 0 to 2π)

$$\begin{gathered}\int \omega_R(\eta)\left[\varkappa_1 \operatorname{ctg}\frac{\eta-\xi}{2} + \sin(\eta+\xi)\right]d\eta - \\ \int \omega_I(\eta)\cos(\eta+\xi)\,d\eta = -2\pi f_{1I}(\xi) \\ -\int \omega_R(\eta)\cos(\eta+\xi)\,d\eta + \\ \int \omega_I(\eta)\left[\varkappa_1 \operatorname{ctg}\frac{\eta-\xi}{2} - \sin(\eta+\xi)\right]d\eta = 2\pi f_{1R}(\xi) \\ \int \omega_R(\eta)\,d\eta = 0, \quad \int \omega_I(\eta)\,d\eta = 0\end{gathered} \tag{6.1}$$

when condition (3.9) is imposed on the right-hand sides. This set of equations has a unique solution for any $\varkappa_1 \neq \pm 1$. The points η_i and ξ_j must be selected as above. This results in the set of linear algebraic equations

$$\begin{gathered}\sum \omega_R(\eta_i)\left[\varkappa_1 \operatorname{ctg}\frac{\eta_i-\xi_j}{2} + \sin(\eta_i+\xi_j)\right]\frac{2\pi}{n} - \\ \sum \omega_I(\eta_i)\cos(\eta_i+\xi_j)\frac{2\pi}{n} + \beta_1 = -2\pi f_{1I}(\xi_j) \\ -\sum \omega_R(\eta_i)\cos(\eta_i+\xi_j)\frac{2\pi}{n} + \\ \sum \omega_I(\eta_i)\left[\varkappa_1 \operatorname{ctg}\frac{\eta_i+\xi_j}{2} - \sin(\eta_j+\xi_j)\right]\frac{2\pi}{n} + \beta_2 = 2\pi f_{1R}(\xi_j) \\ \sum \omega_R(\eta_i)\frac{2\pi}{n} = 0, \quad \sum \omega_I(\eta_i)\frac{2\pi}{n} = 0\end{gathered} \tag{6.2}$$

where the summation is over i from 1 to n.

As regards the regularizing factors β_1 and β_2, the convergence of the solution of Eqs. (6.2) to that of Eqs.(6.1), and also the stability of solution (6.2), statements similar to those made in Sect.5 for the second basic problem, are true in this case.

Problems for any simply connected regions whose contour is smooth with a parametric equation satisfying the conditions described in Sect.2, can be solved similarly. For the solution it is only necessary to know the parametric specification of the contour of the region.

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